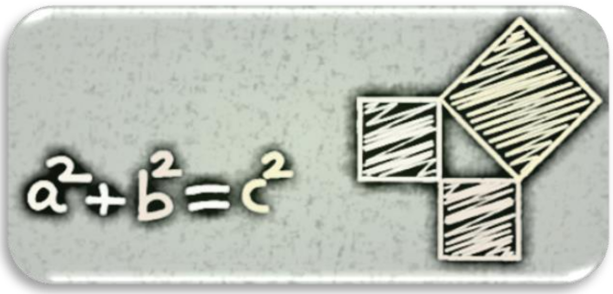
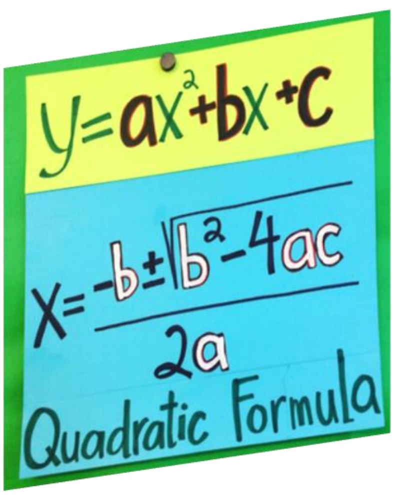
SQUARE ROOTS

You might remember **square roots** if you have studied the Pythagorean Theorem. Perhaps they were hidden, but when the end of a right-triangle problem resulted in an equation like $c^2 = 144$, you used the notion of square root when you determined that c had to be 12. (Another solution to this equation is $c = -12$, but it would be discarded in a triangle problem.)



If you're familiar with The Quadratic Formula, you explicitly saw square roots when you looked at the part of the formula containing the radical sign, $\sqrt{\ }$.



This chapter delves into more detail on square roots, including their precise meaning and how to simplify them. We'll need these skills to continue our study of the Pythagorean Theorem and the Quadratic Formula (which I'm *sure* you're looking forward to).

❑ THE MEANING OF SQUARE ROOTS

The number **81** has two square roots, 9 and –9. This is because $9 \times 9 = 81$, and also $-9 \times -9 = 81$.

Every positive number has two square roots.

Because both 11^2 and $(-11)^2$ equal 121, the two square roots of **121** are 11 and –11.

The number **1** has two square roots, namely 1 and –1. The reason: If either 1 or –1 is squared, the result is 1.
Zero has just one square root, 0, since $0^2 = 0$, and no other number squared would result in 0.

ANY number squared is greater than or equal to zero:

For any *n*,

$$n^2 \geq 0$$

The number **–16** has no square roots. After all, what number squared equals –16? 4 can't work, since $4^2 = +16$. Nor can –4 work, since $(-4)^2$ is also +16. In fact, any number squared produces an answer that is at least zero. That is, no number can be squared and result in –16. Thus, –16 has no square root (until a later course, perhaps).

Consider the fraction $\frac{1}{4}$. It has two square roots, $\frac{1}{2}$ and $-\frac{1}{2}$. Why? I'll tell you why. It's because

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \text{ and also } \left(-\frac{1}{2}\right)^2 = \frac{1}{4}.$$

We can encapsulate this section by stating that

a is a ***square root*** of *b* if $a^2 = b$

Homework

1. a. Find as many numbers as you can whose square is 144.

   b. Find as many numbers as you can whose square is –144.

   c. How many square roots does 25 have? What are they? Why are they square roots of 25?

   d. How many square roots does 196 have? What are they? Why are they square roots of 196?

   e. How many square roots does 1 have? What are they?

   f. How many square roots does 0 have? What are they?

   g. How many square roots does –9 have? What are they?

   h. What are the two square roots of $\frac{4}{9}$?

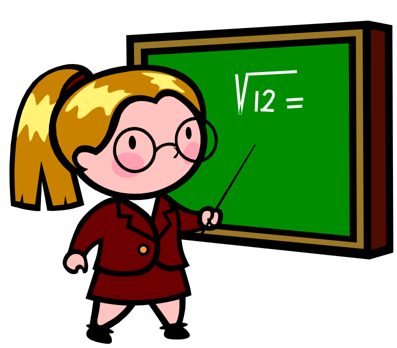
❑ NOTATION

The previous section showed us that the number 81 actually has two square roots, 9 and –9. How do we denote these two square roots of 81 without calculating their actual values? Notice that 81 has a positive square root (the 9) and a negative square root (the –9). We write

$\sqrt{81} = 9$ The positive square root of 81 is 9.

and $-\sqrt{81} = -9$ The negative square root of 81 is –9.

That is, the positive square root is written with a radical sign, while the negative square root is written with a radical sign preceded by a minus sign. The number 121 has two square roots, which we can write as $\sqrt{121} = 11$ and $-\sqrt{121} = -11$.



Thus, if someone asks you for the square root of 100, you must answer 10 and –10, because each of these is a square root of 100. However, if you're presented with " $\sqrt{100}$ ", you answer just 10, <u>not</u> –10, because we agreed that the radical sign refers to only the non-negative square root. Also, if someone asks you for the "non-negative" or ***principal square root*** of 100, then you must answer just 10, not –10.

The square root symbol $\sqrt{}$ is called the ***radical sign***, and the quantity inside the radical sign is called the ***radicand***. Why do we need a radical sign at all? After all, the two square roots of 49 are 7 and –7; it's just that simple. But what about the number 17? It also has two square roots. As you may recall, they don't result in nice numbers (they're infinite, non-repeating decimals), so we must have some notation to denote the two square roots of 17, even though we might not care to work them out. So, we use the radical sign and state that

There are two square roots of 17: $\sqrt{17}$ and $-\sqrt{17}$.

The vital thing to note is that $\left(\sqrt{17}\right)^2 = 17$ and $\left(-\sqrt{17}\right)^2 = 17$.

We state the following four conclusions regarding square roots:

- Every <u>positive</u> number has two square roots, one positive and one negative. The two square roots of n are denoted $\sqrt{n}$ and $-\sqrt{n}$.
- <u>Zero</u> has one square root (namely itself) — $\sqrt{0} = 0$.
- <u>Negative</u> numbers don't have any square roots at all (at least in this course). For example, $\sqrt{-9}$ does not exist.

- <u>The most central issue</u> in this whole discussion is that assuming x is positive (which means neither negative nor zero):

> The symbol $\sqrt{x}$ means the <u>positive</u> square root <u>only</u>!

Homework

> In computers, the *square root* of *n* might be written
>
> **sqrt(n)**

2. a. Name the two square roots of 49. Explain.
   b. Name the two square roots of 29. Explain.
   c. Name the two square roots of 51. Explain.

3. Use your calculator to give some credence to the statement that $(\sqrt{13})^2 = 13$.

4. Evaluate each expression:

a. $\sqrt{144}$	b. $-\sqrt{225}$	c. $\sqrt{-25}$	d. $-\sqrt{-36}$
e. $\sqrt{196}$	f. $\sqrt{1}$	g. $\sqrt{0}$	h. $\sqrt{256}$
i. $-\sqrt{64}$	j. $\sqrt{169}$	k. $\sqrt{-100}$	l. $\sqrt{289}$

5. Simplify each expression:

a. $(\sqrt{64})^2$ b. $(\sqrt{93})^2$ c. $(\sqrt{a})^2$ d. $(\sqrt{m})^2$

6. Consider the following examples:

   i) $\sqrt{9^2} = \sqrt{81} = 9$ ii) $\sqrt{(-6)^2} = \sqrt{36} = 6$

   Now work the following problems (without a calculator):

   a. $\sqrt{10^2}$ b. $\sqrt{12{,}456^2}$ c. $\sqrt{(-7)^2}$ d. $\sqrt{(-3124)^2}$

7. It's easy to confuse the phrases "the negative square root of a number" and "the square root of a negative number." Give examples that clarify the difference between these phrases.

❑ *Simplifying Square Roots*

Let's do an experiment. Our goal is to see if there's a way we can simplify square roots (kind of like we reduce fractions). Specifically, we'll see if we can split a square root into two separate square roots so that one of the square roots will end up being a whole number. That would make the number in the radical smaller than it was, and thus simpler to work with.

For example, let's see what we can do with $\sqrt{36}$. [We know the answer is 6 (and just positive 6, right?), so we'll see if we've accomplished anything useful in our experiment.]

We start with:	$\sqrt{36}$
Factor 36 into 9 times 4:	$\sqrt{9 \cdot 4}$
Now try splitting the radical down the middle:	$\sqrt{9} \cdot \sqrt{4}$
Now calculate the square roots:	$3 \cdot 2$

And finish the arithmetic: 6 ✓

Voilà! It worked. We may have just calculated $\sqrt{36}$ in a crazy long way, but now we have a method to simplify the square root of numbers like 50, which will not come out as a whole number. Let's try it.

EXAMPLE 1: **Simplify: $\sqrt{50}$**

Solution:

We start with: $\sqrt{50}$

Factor 50 into 25 times 2: $\sqrt{25 \cdot 2}$

Split the radical down the middle: $\sqrt{25} \cdot \sqrt{2}$

Now calculate the square root: $5 \cdot \sqrt{2}$

Remove the dot, and we're finished: $\boxed{5\sqrt{2}}$

You might be wondering how we knew to factor 50 as 25 · 2 rather than 10 · 5, perhaps. Well, we could have, but it wouldn't have done us any good because neither 10 nor 5 has a nice square root, whereas 25 does. We say that 25 is a **perfect square** because it has a whole-number square root.

The following is a list of some perfect squares:

4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289

These are the kinds of numbers we seek when we factor the radicand.

We can summarize the success of our experiment by saying that

The square root of a product is equal to the product of their square roots.

Thus, assuming that a and b are both *non-negative* (that means at least zero),

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b}$$

The square root of a product is equal to the product of their square roots.

EXAMPLE 2: **Simplify each square root:**

A. $\sqrt{144} = \sqrt{36\cdot 4} = \sqrt{36}\cdot\sqrt{4} = 6\cdot 2 = \mathbf{12}$

(Of course, the steps aren't needed, since $\sqrt{144}$ is easily seen to be 12, but it again proves that the method works.)

B. $\sqrt{75} = \sqrt{25\cdot 3} = \sqrt{25}\cdot\sqrt{3} = \mathbf{5\sqrt{3}}$ (25 is a perfect square)

C. $\sqrt{40} = \sqrt{4\cdot 10} = \sqrt{4}\cdot\sqrt{10} = \mathbf{2\sqrt{10}}$ (4 is a perfect square)

D. $\sqrt{45} = \sqrt{9\cdot 5} = \sqrt{9}\cdot\sqrt{5} = \mathbf{3\sqrt{5}}$ (9 is a perfect square)

E. $-\sqrt{288} = -\sqrt{144\cdot 2} = -\sqrt{144}\cdot\sqrt{2} = \mathbf{-12\sqrt{2}}$

(144 is a perfect square)

F. $\sqrt{72} = \sqrt{36\cdot 2} = \sqrt{36}\cdot\sqrt{2} = \mathbf{6\sqrt{2}}$ (36 is a perfect square)

This problem could have been done in more steps; you are encouraged to use as many steps as you see fit (unless your teacher thinks otherwise). For example,

$$\sqrt{72} = \sqrt{9\cdot 8} = \sqrt{9}\cdot\sqrt{8} = 3\sqrt{8} = 3\sqrt{4\cdot 2}$$

$$= 3\cdot\sqrt{4}\cdot\sqrt{2} = 3\cdot 2\cdot\sqrt{2} = \mathbf{6\sqrt{2}}\text{, the same answer}$$

It's important to understand that if you had stopped at the step $3\sqrt{8}$, you would not have the right answer, even though you simplified the radical somewhat. This is because even more factors could be brought out of the square root sign.

G. $\sqrt{30}$ cannot be simplified. 30 does not contain any perfect-square factors.

H. $\sqrt{-64}$ does not exist in Algebra 1.

Homework

8. In part F of the above example, we concluded that $\sqrt{72} = 6\sqrt{2}$. Use your calculator to "verify" this fact.

9. Simplify each square root:

a. $\sqrt{8}$	b. $\sqrt{24}$	c. $\sqrt{-75}$	d. $\sqrt{27}$	e. $\sqrt{98}$
f. $\sqrt{54}$	g. $\sqrt{150}$	h. $\sqrt{172}$	i. $\sqrt{105}$	j. $\sqrt{-20}$
k. $\sqrt{175}$	l. $\sqrt{200}$	m. $\sqrt{242}$	n. $\sqrt{46}$	o. $\sqrt{-1}$
p. $\sqrt{1}$	q. $\sqrt{0}$	r. $\sqrt{88}$	s. $\sqrt{147}$	t. $\sqrt{361}$
u. $\sqrt{338}$	v. $\sqrt{605}$	w. $\sqrt{250}$	x. $\sqrt{184}$	y. $\sqrt{189}$

10. Simplify each square root:

a. $\sqrt{18}$	b. $-\sqrt{72}$	c. $\sqrt{12}$	d. $\sqrt{32}$	e. $-\sqrt{52}$
f. $\sqrt{162}$	g. $\sqrt{125}$	h. $\sqrt{128}$	i. $\sqrt{256}$	j. $\sqrt{28}$
k. $\sqrt{-9}$	l. $-\sqrt{500}$	m. $\sqrt{289}$	n. $\sqrt{625}$	o. $\sqrt{800}$
p. $\sqrt{600}$	q. $\sqrt{-4}$	r. $\sqrt{448}$	s. $-\sqrt{121}$	t. $\sqrt{450}$
u. $\sqrt{363}$	v. $\sqrt{117}$	w. $-\sqrt{350}$	x. $\sqrt{490}$	y. $\sqrt{396}$

11. We have a pretty powerful rule, $\sqrt{ab} = \sqrt{a}\sqrt{b}$, that we've been using to simplify square roots. A natural question might be: Does it also work for addition? That is, is it always true that $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$? Test this theory yourself using the values $a = 16$ and $b = 9$.

Review Problems

12. a. How many square roots does 169 have? What are they? Why are they square roots of 169?

    b. How many square roots does 29 have? What are they? Why are they square roots of 29?

    c. Name the only number with exactly one square root.

    d. Give a number that has no square root in Elementary Algebra.

    e. What is the smallest possible value of the expression $\sqrt{x}$?

    f. Consider the number $-\sqrt{-36}$. Your uncle tells you the answer is 6, since the negatives cancel out. Why is your uncle ignorant of math? Should you tell him that he is?

g. Simplify: $\left(\sqrt{n}\right)^2$ (presuming n is a non-negative number.)

h. Calculate $\sqrt{123{,}987{,}005^2}$ without a calculator.

i. Calculate $\sqrt{(-1{,}273)^2}$ without a calculator.

13. Simplify each square root:

a. $\sqrt{250}$ b. $\sqrt{56}$ c. $\sqrt{112}$ d. $\sqrt{400}$ e. $\sqrt{144}$

f. $\sqrt{76}$ g. $-\sqrt{288}$ h. $\sqrt{8}$ i. $\sqrt{4}$ j. $-\sqrt{54}$

k. $\sqrt{300}$ l. $\sqrt{9}$ m. $\sqrt{49}$ n. $\sqrt{196}$ o. $\sqrt{475}$

p. $\sqrt{72}$ q. $\sqrt{240}$ r. $\sqrt{100}$ s. $\sqrt{52}$ t. $\sqrt{96}$

u. $\sqrt{98}$ v. $-\sqrt{2}$ w. $\sqrt{648}$ x. $\sqrt{500}$ y. $\sqrt{-3}$

14. Simplify each square root:

a. $\sqrt{275}$ b. $\sqrt{931}$ c. $\sqrt{208}$ d. $\sqrt{343}$ e. $-\sqrt{63}$

f. $-\sqrt{175}$ g. $\sqrt{360}$ h. $\sqrt{44}$ i. $\sqrt{375}$ j. $\sqrt{68}$

k. $\sqrt{810}$ l. $\sqrt{80}$ m. $\sqrt{99}$ n. $\sqrt{252}$ o. $\sqrt{160}$

p. $\sqrt{153}$ q. $-\sqrt{180}$ r. $\sqrt{588}$ s. $\sqrt{135}$ t. $\sqrt{192}$

u. $\sqrt{425}$ v. $\sqrt{176}$ w. $\sqrt{92}$ x. $\sqrt{720}$ y. $\sqrt{-60}$

Solutions

1. a. 12 and –12 each have squares equal to 144.

b. There's no number (in this class) whose square is –144.

c. 25 has two square roots, 5 and –5 (which we could write as ±5).

5 is a square root of 25 because $5^2 = 25$.

–5 is a square root of 25 because $(-5)^2 = 25$.

d. 196 has two square roots, 14 and –14.

14 is a square root of 196 because $14^2 = 196$.

–14 is a square root of 196 because $(-14)^2 = 196$.

e. 1 has two square roots, 1 and –1.

f. 0 has only one square root, 0.

g. –9 has no square root at all (but it might in Intermediate Algebra!).

h. The two square roots of $\frac{4}{9}$ are $\frac{2}{3}$ and $-\frac{2}{3}$.

2. a. $\sqrt{49}$ and $-\sqrt{49}$, OR 7 and –7 because $7^2 = 49$ and $(-7)^2 = 49$

b. $\sqrt{29}$ and $-\sqrt{29}$ because $(\sqrt{29})^2 = 29$ and $(-\sqrt{29})^2 = 29$

c. $\sqrt{51}$ and $-\sqrt{51}$ because $(\sqrt{51})^2 = 51$ and $(-\sqrt{51})^2 = 51$

3. $(\sqrt{13})^2 \approx (3.605551275)^2 \approx 13$

4.

a. 12	b. –15	c. Does not exist
d. Does not exist	e. 14	f. 1
g. 0	h. 16	i. –8
j. 13	k. Does not exist	l. 17

5. a. 64 b. 93 c. a d. m

6. a. 10 b. 12,456 c. 7 d. 3124

7. An example of "*the negative square root of a number*" is $-\sqrt{100}$, which comes out to be –10, a fine answer.

As for "*the square root of a negative number*," an example would be $\sqrt{-1}$, which we've learned is not even a number in Elementary Algebra.

8. $\sqrt{72} \approx 8.48528$ and $6\sqrt{2} \approx 6(1.41421) = 8.48526$

Did we get the same decimal result? Not exactly, but this was to be expected because of rounding. Nevertheless, this calculation gives pretty good evidence that $\sqrt{72}$ truly equals $6\sqrt{2}$. In fact, if you don't do any rounding at all — using all the digits your calculator can handle — the two quantities might have exactly the same digits.

9.

a. $2\sqrt{2}$	b. $2\sqrt{6}$	c. Does not exist	d. $3\sqrt{3}$	e. $7\sqrt{2}$
f. $3\sqrt{6}$	g. $5\sqrt{6}$	h. $2\sqrt{43}$	i. $\sqrt{105}$	j. Does not exist
k. $5\sqrt{7}$	l. $10\sqrt{2}$	m. $11\sqrt{2}$	n. $\sqrt{46}$	o. Does not exist
p. 1	q. 0	r. $2\sqrt{22}$	s. $7\sqrt{3}$	t. 19
u. $13\sqrt{2}$	v. $11\sqrt{5}$	w. $5\sqrt{10}$	x. $2\sqrt{46}$	y. $3\sqrt{21}$

10.

a. $3\sqrt{2}$	b. $-6\sqrt{2}$	c. $2\sqrt{3}$	d. $4\sqrt{2}$	e. $-2\sqrt{13}$
f. $9\sqrt{2}$	g. $5\sqrt{5}$	h. $8\sqrt{2}$	i. 16	j. $2\sqrt{7}$
k. Does not exist	l. $-10\sqrt{5}$	m. 17	n. 25	o. $20\sqrt{2}$
p. $10\sqrt{6}$	q. Does not exist	r. $8\sqrt{7}$	s. -11	t. $15\sqrt{2}$
u. $11\sqrt{3}$	v. $3\sqrt{13}$	w. $-5\sqrt{14}$	x. $7\sqrt{10}$	y. $6\sqrt{11}$

11. Using $a = 16$ and $b = 9$ as test numbers:

$$\sqrt{a+b} = \sqrt{16+9} = \sqrt{25} = 5$$

$$\sqrt{a}+\sqrt{b} = \sqrt{16}+\sqrt{9} = 4+3 = 7$$

Not the same result! Our theory that $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ is bogus (we have found a counterexample). It's also false if we use subtraction, but it's true if we use division (as long as we never divide by 0).

12. a. 169 has two square roots, 13 and –13 (or ± 13). They are square roots of 169 because $13^2 = 169$ and $(-13)^2 = 169$.

b. 29 has two square roots, $\sqrt{29}$ and $-\sqrt{29}$ (or $\pm\sqrt{29}$). They are square roots of 29 because $\left(\sqrt{29}\right)^2 = 29$ and $\left(-\sqrt{29}\right)^2 = 29$.

c. 0; its only square root is 0.

d. –49, for instance, has no square root (in this class). In fact, no negative number has a square root in this class.

e. The smallest possible value is 0, and that occurs when $x = 0$.

f. The negatives do <u>not</u> cancel out; they're not right next to each other. The Order of Operations requires that we start with the square root, which doesn't exist because the radicand is negative. Should you inform your uncle that he's ignorant of math? Not if you hope to be in his will.

g. n h. 123,987,005 i. 1,273

13.

a. $5\sqrt{10}$	b. $2\sqrt{14}$	c. $4\sqrt{7}$	d. 20	e. 12
f. $2\sqrt{19}$	g. $-12\sqrt{2}$	h. $2\sqrt{2}$	i. 2	j. $-3\sqrt{6}$
k. $10\sqrt{3}$	l. 3	m. 7	n. 14	o. $5\sqrt{19}$
p. $6\sqrt{2}$	q. $4\sqrt{15}$	r. 10	s. $2\sqrt{13}$	t. $4\sqrt{6}$
u. $7\sqrt{2}$	v. $-\sqrt{2}$	w. $18\sqrt{2}$	x. $10\sqrt{5}$	y. Does not exist

14.

a. $5\sqrt{11}$	b. $7\sqrt{19}$	c. $4\sqrt{13}$	d. $7\sqrt{7}$	e. $-3\sqrt{7}$
f. $-5\sqrt{7}$	g. $6\sqrt{10}$	h. $2\sqrt{11}$	i. $5\sqrt{15}$	j. $2\sqrt{17}$
k. $9\sqrt{10}$	l. $4\sqrt{5}$	m. $3\sqrt{11}$	n. $6\sqrt{7}$	o. $4\sqrt{10}$
p. $3\sqrt{17}$	q. $-6\sqrt{5}$	r. $14\sqrt{3}$	s. $3\sqrt{15}$	t. $8\sqrt{3}$
u. $5\sqrt{17}$	v. $4\sqrt{11}$	w. $2\sqrt{23}$	x. $12\sqrt{5}$	y. Does not exist

"A man does what he must – in spite of personal consequences, in spite of obstacles and dangers and pressures – and that is the basis of all human morality."

John F. Kennedy (1917–1963)